INTRODUCTION BASICS & AXIOMS OF QM & APPLICATION TO TWO LEVEL SYSTEMS (qubits)

Formulated here as purely mathematical axioms.

(as first done probably by von Neumann historically).

As we go we will see physical examples of systems to apply them.

Maths needed for us: Lin alg, matrix theory, complex numbers. (In more advanced settings → funct analysis).

1) A few Math recaps & Dirac Notation (Prelim).

Complex Hilbert space $\mathcal{H}$ = vector space on a complex field $\mathbb{C}$. Vectors denoted $|\psi\rangle$, $|\chi\rangle$, ..., scalar #s noted $\alpha, \beta, \lambda, \mu \in \mathbb{C}$.

Transpose vectors $(|\psi\rangle)^{T,*} = \langle\psi|$, $(|\chi\rangle)^{T,*} = \langle\chi|$.

Scalar product or Dirac bracket $\langle\psi|\chi\rangle \in \mathbb{C}$.

skew sym: $\overline{\langle \psi | \chi \rangle} = \langle \chi | \psi \rangle$.

bilinear: $\langle \psi | (\alpha | \chi_1 \rangle + \beta | \chi_2 \rangle)$

$$= \alpha \langle \psi | \chi_1 \rangle + \beta \langle \psi | \chi_2 \rangle$$

$$(\bar{\alpha} \langle \varphi_1 | + \bar{\beta} \langle \varphi_2 |) | \psi \rangle$$

$$= \bar{\alpha} \langle \varphi_1 | \psi \rangle + \bar{\beta} \langle \varphi_2 | \psi \rangle.$$

positive: $\langle \psi | \psi \rangle \geq 0$ and equality $\Rightarrow | \psi \rangle = 0$.

Properties: $\langle \psi | \psi \rangle = \| \psi \|^2$ defines a norm.

$$\sqrt{\langle \psi | \psi \rangle} = \| \psi \|$$

triangle ineq. $\| \psi + \chi \| \leq \| \psi \| + \| \chi \|$

Cauchy Schwarz $| \langle \chi | \psi \rangle | \leq \| \chi \| \| \psi \|$.

Examples of Hilbert space:

- $\mathbb{C}^2$ $\quad \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ product $\quad \bar{\alpha} \langle 0 | + \bar{\beta} \langle 1 | = (\bar{\alpha} \ \bar{\beta})$
- $\mathbb{C}^d$ $\quad \sum_{i=1}^{d} \alpha_i | i \rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}$ product. $\quad$ Euclidean inner prod. $\bar{\alpha} \gamma + \bar{\beta} \delta$.
- $L^2(\mathbb{R}^3) \ni \psi(\underline{x}), \ \underline{x} \in \mathbb{R}^3$ $\quad \int | \psi(x) |^2$
  $= \langle x | \psi \rangle$ $\quad$ inner product $\int \overline{\varphi(x)} \chi(x) = \langle \varphi | \chi \rangle$ $= \int dx \langle \overline{\varphi(x)} \rangle \langle \chi(x) \rangle$

Tensor product. For 2 Hilbert space $\mathcal{H}_1$, $\mathcal{H}_2$

we can form a bigger one $\mathcal{H}_1 \otimes \mathcal{H}_2$ $(= \mathcal{H}_{1\cup 2})$

Concretely with basis $|v_1\rangle, \dots |v_{d_1}\rangle$

$|w_1\rangle, \dots |w_{d_2}\rangle$

$\mathcal{H}_1 \otimes \mathcal{H}_2$ has basis made of all pairings $d_1 d_2$ $|v_i\rangle \otimes |w_j\rangle = |v_i w_j\rangle$

Dimension is $d_1 d_2$. Elements $|\psi\rangle = \sum_{ij=1} \alpha_{ij} |v_i w_j\rangle$

$\langle\psi| = \sum \bar{\alpha}_{ij} \langle v_i w_j|$

Inner product $\langle v_i w_j | v_\ell w_k\rangle$

$= \langle v_i | v_\ell\rangle \langle w_j | w_k\rangle = \delta_{i\ell} \delta_{jk}$

orthonormal

in general: $(\langle\chi_1| \otimes \langle\chi_2|)(|\psi_1\rangle \otimes |\psi_2\rangle)$

$= \langle\chi_1|\psi_1\rangle \langle\chi_2|\psi_2\rangle.$

Def: Product vector (state) if $|\psi\rangle$ can be written as $|\chi_1\rangle \otimes |\chi_2\rangle$. **Note** $\alpha_{ij} = \beta_i \gamma_j \Rightarrow |\psi\rangle$ is product vector.

Examples $\mathbb{C}^2 \otimes \mathbb{C}^2, \ldots \quad L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.

Matrices. let $\mathcal{H}$ of dimension d.

square matrix $A : \mathcal{H} \to \mathcal{H}$

$$|\psi\rangle \mapsto A|\psi\rangle .$$

is a linear map $A(|\psi\rangle + |\chi\rangle) = A|\psi\rangle + A|\chi\rangle$.

Representation in a basis $\{|1\rangle, \ldots |d\rangle\}$ $|i\rangle$.

by Tableau. (orthonormal basis)

$$A_{ij} = \langle i | A | j \rangle \in \mathbb{C}$$

(for otho basis).

Property: $A = \sum_{i,j} A_{ij} |i\rangle\langle j|$

Proof: $\sum_{i,j} \langle i | A | j\rangle |i\rangle\langle j| = \sum_{i,j} \underbrace{|i\rangle\langle i|}_{\text{project on vect } |i\rangle} A \underbrace{|j\rangle\langle j|}_{\text{project on vect } |j\rangle}$

$= A$

$\sum_i |i\rangle\langle i| = \mathbb{1}$ & $\sum_j |j\rangle\langle j| = \mathbb{1}$

(called the closure relation)

Tensor product of matrices.

$$A : \mathcal{H}_1 \to \mathcal{H}_1 . \qquad d_1 \times d_1$$

$$B : \mathcal{H}_2 \to \mathcal{H}_2 . \qquad d_2 \times d_2 .$$

$$(A \otimes B) : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 .$$

$$|\psi_1\rangle \otimes |\psi_2\rangle \to (A \otimes B)|\psi_1\rangle \otimes |\psi_2\rangle$$

$$= A|\psi_1\rangle \otimes B|\psi_2\rangle .$$

(+ linearity to complete definition).

Tableau repr.:

$$\langle i, j | A \otimes B | k, l\rangle = \langle i, j | (A|k\rangle \otimes B|l\rangle)$$

$$= \langle i | A | k\rangle \langle j | B | l\rangle$$

$$= A_{ik} B_{jl}$$

$$= (A \otimes B)_{ij;\, kl} . \qquad d_1 d_2 \times d_1 d_2 .$$

Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \end{pmatrix} \qquad 4 \times 4$$

Property:

$$A \otimes B = \sum_{ij} A_{ij} |i\rangle\langle j| \otimes \sum_{kl} B_{kl} |k\rangle\langle l|$$

$$= \sum_{ijkl} A_{ij} B_{kl} |i\,k\rangle\langle j\,l|.$$

Hermitian matrices: $A = A^{T,*} = A^{\dagger}$.

$$A_{ij} = A^{*}_{ji}$$

$$\langle i|A|j\rangle = \overline{\langle j|A|i\rangle}$$

Unitary matrices. $U^{\dagger}U = UU^{\dagger} = \mathbb{1}$.

prop: · lines are orthonormal basis
· col are orthonormal basis.
· $U^{-1} = U^{\dagger}$.

(Exercise.)

· preserve sc prod & norm (⇔ unitarity).

$$(U|\psi\rangle)^{T,*}(U|\chi\rangle) = \langle\psi|U^{\dagger}U|\chi\rangle$$

$$= \langle\psi|U^{-1}U|\chi\rangle$$

$$= \langle\psi|\chi\rangle$$

· $\|U\psi\| = \|\psi\|$.

2) Axioms of QM.

I. An isolated syst is described by a state vector element of a Hilbert space $\mathcal{H}$. $|\psi\rangle$. We must have $\|\psi\| = 1$.

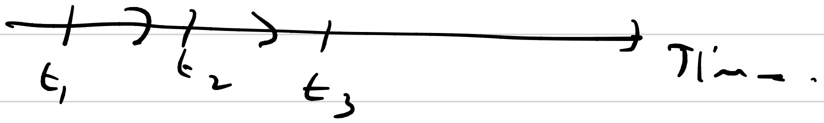
II. The time evolution of the state vector is given by a unitary matrix:

$$|\psi(t)\rangle = U_t \, |\psi(0)\rangle.$$

where $U_t^\dagger U_t = U_t U_t^\dagger = \mathbb{1}$.

In particular $\|\psi(t)\| = \|\psi(0)\| = 1$. The matrices satisfy composition property:

$$U(t_3, t_2)\, U(t_2, t_1) = U(t_3, t_1).$$



III. <u>Observable quantities</u> are represented by hermitian matrices A (i.e s.t $A = A^\dagger$).

IV. <u>Meas postulate</u>. a) If a system is a state $|\psi\rangle$ and an observable A is measured then the result of the Meas is $\underbrace{\text{an}}_{\text{random}}$ eigenvalue α_i of A $\{i = 1 \ldots d\}$ with probability $|\langle i|\psi\rangle|^2$ when $|i\rangle$ is eigenvector and state after Meas is $|i\rangle$.

<u>von Neumann Projective Measurements</u>

b) The measurement apparatus for us is equivalent to a choice $\underbrace{\text{of}}_{\text{orth}}$ basis $\{|1\rangle, \ldots |d\rangle\}$. With this Meas App we can measure any obs that has these eigenvectors. If state is $|\psi\rangle$ the state after meas is $|i\rangle$ with prob $|\langle i|\psi\rangle|^2 = \text{prob}(i)$.

Remark: $\{ |i\rangle\langle i| = \Pi_i$ are proj $\int$ apparatus. $\sum_i \Pi_i = \mathbb{1}$.

$$\text{prob}(i) = \langle \psi | \Pi_i | \psi \rangle = |\langle i|\psi\rangle|^2.$$

state after Meas $\quad \Pi_i|\psi\rangle / \|\Pi_i|\psi\rangle\|$.

<u>Remark</u>

- $$\sum_{i=1}^{d} \text{prob}(i) = \sum_{i=1}^{d} |\langle i|\psi\rangle|^2$$
$$= \sum_{i=1}^{d} \langle\psi|i\rangle\langle i|\psi\rangle = \langle\psi|\psi\rangle$$
$$= 1.$$

- If $|\psi(0)\rangle = |\psi\rangle$ and we let syst evolve for a time t then measure $|\psi(t)\rangle$ with apparatus $\{|1\rangle \dots |d\rangle\}$ the syst after Meas is $|i\rangle$ with prob
$$|\langle i|U_t|\psi\rangle|^2 = \langle\psi|U_t^+ \Pi_i U_t|\psi\rangle$$

Such bracket. Often called a "transition probability"

V. Composition of systems.

Syst A described by $\mathcal{H}_A$, Syst B desc by $\mathcal{H}_B$.

Syst $A \cup B$ described by $\mathcal{H}_A \otimes \mathcal{H}_B$.

State $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

<u>Def</u> Product & Entangled state.

- $|\psi\rangle$ is a product state if $\exists\, |\chi_A\rangle \in \mathcal{H}_A$ and $|\chi_B\rangle \in \mathcal{H}_B$ s.t. $|\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$.
- $|\psi\rangle$ is an entangled state if it is not product.

<u>Property</u> If syst is in state $|\psi\rangle$ and A is measured the statistics of this meas satisfy:

exp value of $A = \sum_i \alpha_i |\langle i|\psi\rangle|^2 = \langle\psi|A|\psi\rangle$.

var of $A = \sum \alpha_i^2 |\langle i|\psi\rangle|^2 - \langle\psi|A|\psi\rangle^2$

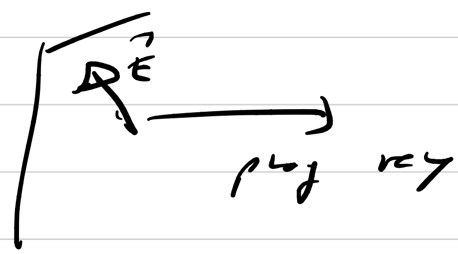
$= \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$.

(Exercise).

3) Qubits & ensembles of qubits.

Many physical systems are constituted by assemblies of discrete 2-dim systems whose individual Hilbert space is $\mathcal{H} = \mathbb{C}^2$. For N such 2-dim systs $\mathcal{H}_{total} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N \text{ times}}$.

Examples: 1) pol state of a photon is a vector analogous to $\vec{E}$ field of an e-m wave



in plane $\perp$ to wave vector: two dim basis

But this has complex components

$|V\rangle, |H\rangle$ basis

$|R\rangle, |L\rangle$ dual basis.

$$|\text{pol state}\rangle = \alpha|V\rangle + \beta|H\rangle = \gamma|R\rangle + \delta|L\rangle$$

2) Spin 1/2 : Magn moment of particles such as electron, proton, some nuclei, atoms etc... which take only two possible values when measured. More on this next time.

$$\{ |\uparrow\rangle, |\downarrow\rangle \} ; \{ |\rightarrow\rangle, |\leftarrow\rangle \} ; \ldots$$

3) two level systs

Atoms, ions —— first exc state } can be

—— Gr. state } isolated from rest of spectrum.

- All these constitute natural "qubits".

qubits a two level systs can also be engineered

example superconducting circuits

- qubit turns out to be the elementary natural unit of quantum information. Much as classical bit is unit of classical information.

- $\mathbb{C}^2 = \{ \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad \alpha, \beta \in \mathbb{C} \}$.

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ is an orth basis

computational basis or Z-basis

$\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

is another basis Hadamard basis or X-basis

Other important basis Y-basis!

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

<u>exercise</u>: check orthonormality of Y-basis!

- <u>Multiple qubits</u> : $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

comp basis $\quad |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_N\rangle \quad b_i = 0, 1$

$|\psi\rangle = \sum c_{b_1 \cdots b_N} |b_1 b_2 \cdots b_N\rangle \qquad$ <u>2^N dimensional!</u>

→ Note 2^N classical bit strings can be encoded in N qubits! But all a continuum set of complex nbs.

→ However if we apply the Meas principle we see that a Meas (say in comp basis) allows to extract only one string

$$\mathrm{prob}(b_1 \ldots b_N) = |\langle b_1 \ldots b_N | \psi \rangle|^2$$
$$= |C_{b_1 \ldots b_N}|^2 .$$

→ So one should not think that $\alpha|0\rangle + \beta|1\rangle$ contains an infinite amount of information. How to quantify this is given by von-Neumann entropy (see later on).

4 Bloch sphere: geometric illustration of qubits in $\mathbb{C}^2$.

A qubit state $\alpha|0\rangle + \beta|1\rangle$ has $\alpha, \beta \in \mathbb{C}^2$ with $|\alpha|^2 + |\beta|^2 = \alpha\bar{\alpha} + \beta\bar{\beta} = 1$

There are thus $4 - 1 = 3$ real parameters that matter. In fact in QM states

$$\alpha|0\rangle + \beta|1\rangle \quad , \quad e^{i\lambda}(\alpha|0\rangle + \beta|1\rangle)$$

are equivalent because in any proj meas the global phase λ is not observable.

Indeed $\quad \text{prob}(v) = |\langle v|(e^{i\lambda}\alpha|0\rangle + \beta|1\rangle)|^2$

$$= \underbrace{|e^{i\lambda}|^2}_{1} |\langle v|(\alpha|0\rangle + \beta|1\rangle|^2$$

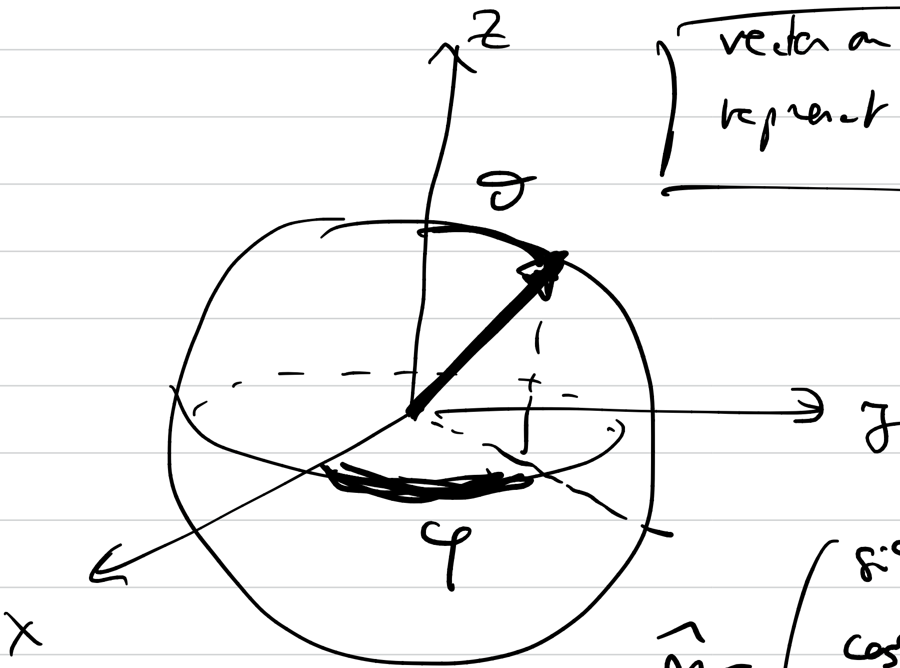
Thus we can choose λ so that $e^{i\lambda}\alpha$ is real.

As a real we need only 2 real params

Standard parametrization is as follows:

$$\alpha = \cos\frac{\vartheta}{2} \qquad \beta = e^{i\varphi}\sin\frac{\vartheta}{2}.$$

$$\begin{cases} |\alpha|^2 + |\beta|^2 = \cos^2\frac{\vartheta}{2} + |e^{i\varphi}|^2 \sin^2\frac{\vartheta}{2} = 1 \\ \alpha \in \mathbb{R}. \end{cases}$$

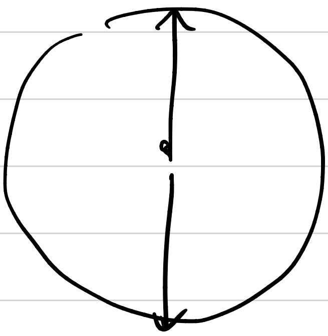


vector on sphere represent the qubit

$$\hat{n}_{\vartheta,\varphi} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \cos\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

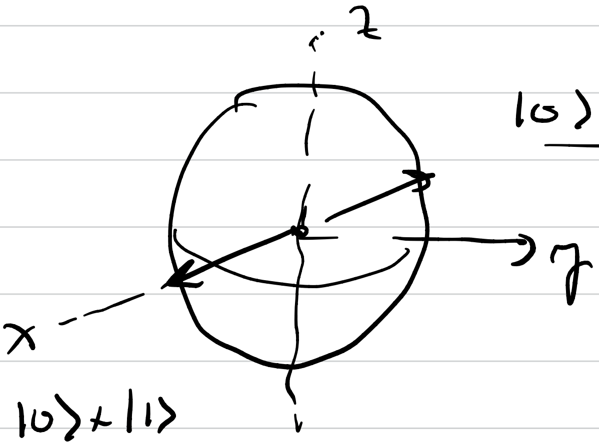
Special cases:

$|0\rangle = |\uparrow\rangle$ $\qquad \vartheta = 0$, φ anything. $\begin{pmatrix}1\\0\\0\end{pmatrix}$



$|1\rangle = |\downarrow\rangle$ $\qquad \vartheta = \pi$, φ anything $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

z-basis $\{ |0\rangle, \ |1\rangle \}$.

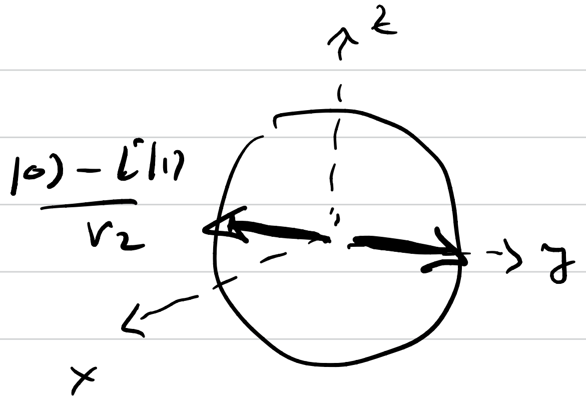


$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ $\qquad \vartheta = \frac{\pi}{2}$; $\varphi = \pi$; $\frac{1}{\sqrt{2}}\begin{pmatrix}\cancel{0}\\1\\0\end{pmatrix}$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

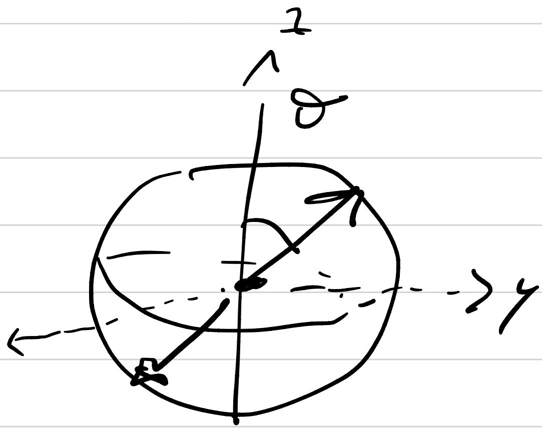
$\vartheta = \frac{\pi}{2}$; $\varphi = 0$; $\frac{1}{\sqrt{2}}\begin{pmatrix}\cancel{0}\\-\cancel{1}\\0\end{pmatrix}$.

X-basis. $\left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}} \ ; \ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\}$



$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad \begin{cases} \theta = \frac{\pi}{2} \; ; \; \varphi = \frac{\pi}{2} \\ \theta = \frac{\pi}{2} \; ; \; \varphi = -\frac{\pi}{2}. \end{cases}$$

Y- basis.



<u>Note</u> on the Bloch sphere

opposite vects

are orthonormal in Hilbert space.

5 Elementary observables for qubits

$A : \mathbb{C}^2 \to \mathbb{C}^2$ $\quad 2\times 2$ hermitian matrices

$A = A^\dagger$.

basis of this space of matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = a_0 \mathbb{1} + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$$ Most general observable.

$a_0, a_x, a_y, a_z \in \mathbb{R}$. $\quad A = A^\dagger$.

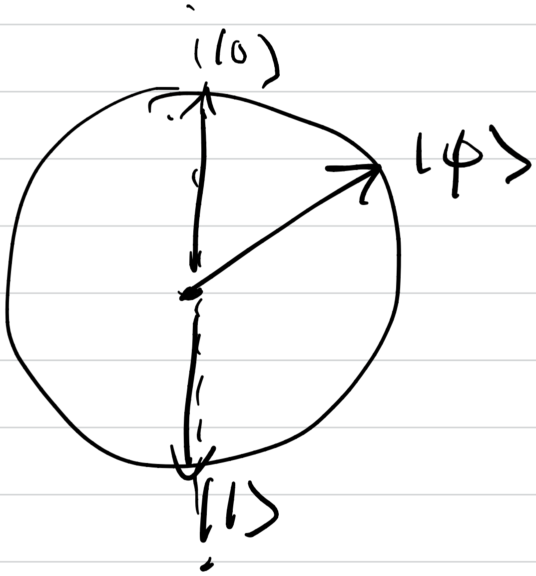
[Note that $\sigma_x, \sigma_y, \sigma_z$ are also unitary but this is a 'coincidence'].

Measurement of σ_z

Eigenvectors are $|0\rangle$, $|1\rangle$ $\Rightarrow$ take Z-basis.

Meas results $|0\rangle$: $p(0) = |\langle 0|\psi\rangle|^2 = \left(\cos\frac{\theta}{2}\right)^2$

$|1\rangle$: $p(1) = |\langle 1|\psi\rangle|^2 = \left(\sin\frac{\theta}{2}\right)^2$.



Measurement projects $|\psi\rangle$ onto $|0\rangle$ or $|1\rangle$ w. prob $\left(\cos\frac{\theta}{2}\right)^2$ or $\left(\sin\frac{\theta}{2}\right)^2$.

#.

<u>Meas of σ_x</u> eigenvectors $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$; $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$\Rightarrow$ take X-basis.

Meas result $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}} = |\pm\rangle$

$$p(\pm) = \left| \left(\frac{\langle 0| \pm \langle 1|}{\sqrt{2}} \right) \left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \right) \right|^2$$

$$= \frac{1}{2} \left| \cos\frac{\theta}{2} \pm e^{i\varphi}\sin\frac{\theta}{2} \right|^2$$

<u>Meas of σ_y</u> exercise.

6 Unitary dynamics in $\mathbb{C}^2$.

Stones thm: any unitary matrix can be written $U = e^{iA}$ where A is hermitian.

Since we must have $U(t_3, t_2)\, U(t_2, t_1) = U(t_3, t_1)$

It follows any unitary dynamics in $\mathbb{C}^2$ is of this form above.

Special case $U_t = e^{-i\frac{tH}{\hbar}}$ with $H = H^\dagger$

and H independent of time.

H is called the (generator or) Hamiltonian in physics.

Most general Hamiltonian is of form

$$H = \vec{B} \cdot \vec{\sigma} = b_x \sigma_x + b_y \sigma_y + b_z \sigma_z$$

$\hbar = [\text{Joule} \cdot \text{sec}] = \frac{h}{2\pi} = \text{Planck constant}/2\pi$.

Remark: Note $a_0 \mathbb{I}$ generates $e^{it\mathbb{I}} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}$

$$= e^{it} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

= a trivial global phase.

Remark: In the case when the qubit is a physical spin 1/2: $\vec{B}$ = magnetic field

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ $\propto$ Magnetic moment observable.

More on this later.

Remark: $i\hbar \frac{d}{dt} U_t = H U_t$; multiply by $|\psi(0)\rangle$

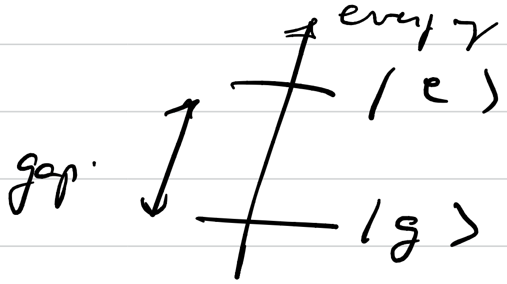
$$\Rightarrow \boxed{i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle} \quad \text{Schroedinger equation.}$$

It turns out this equation is still valid if H depends on time: $\boxed{i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle}$

7 Two level systems, Spin 1/2, Dynamics.

In physics a two level system is any system where you can isolate two orthonormal states which form the basis of the Hilbert space. Typically we will call them

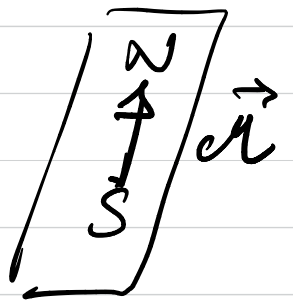
$$\begin{cases} |0\rangle, \ |1\rangle & \text{(qubit interpretation)} \\ |\uparrow\rangle, \ |\downarrow\rangle & \text{(spin 1/2 interpretation)} \\ |g\rangle, \ |e\rangle & \text{(atomic interpretation)} \end{cases}$$



The most general Hamiltonian and unitary dynamics is the one given before.

Spin 1/2 example.

Magnetic moments are orientational, "vector", degrees of freedom reacting to magnetic fields. For example the needle of a compass has a magnetic moment which aligns with the earth's magnetic field. In general magnets has a North and South pole, i.e they are "dipoles" which carry a magnetic moment $\vec{\mathcal{M}}$.



When placed in a magnetic field $\vec{B}$ the "dipole" $\vec{\mathcal{M}}$ will tend to align (or antialign) and minimize an energy $E \propto -\vec{\mathcal{M}}\cdot\vec{B}$

The origin of $\vec{\mathcal{M}}$ (or magnetic moments) is a complicated story (there are various contributions).

microscopic

But in particular in nature some systems (electrons, protons, ...) have an intrinsic magnetic moment which is a "quantum observable".

In other words $\vec{\mathcal{M}} = (\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)$ has components which are hermitian matrices. These may have any integer dimension ≥ 2 depending on the type of magnetic moment.

A particular type of magnetic moment has 2×2 matrix components given by the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

i.e $$\vec{M} \propto \underbrace{\vec{\Sigma}}_{\text{just other notation.}} = (\sigma_x, \sigma_y, \sigma_z)$$

up to proportionality constant g.

The "energy observable" of these magnetic moments is

$$H = -g\,\vec{\Sigma}\cdot\vec{B}$$

magnetic field

gyromagnetic factor

spin 1/2 vector

$$H = -g(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z).$$

$$= -g \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}.$$

<u>Remark</u>: As said before this mathematical structure generates the most general unitary dynamics in $\mathbb{C}^2$ and is mathematically applicable to any two-level system.

Dynamics of "spin" in constant "magnetic field."

If $\vec{B}$ is constant we can choose axis so that $\vec{B} \parallel z$ i.e $\vec{B} = (0, 0, B_z)$.

$$\Rightarrow \qquad H = -g\, B_z \sigma_z$$

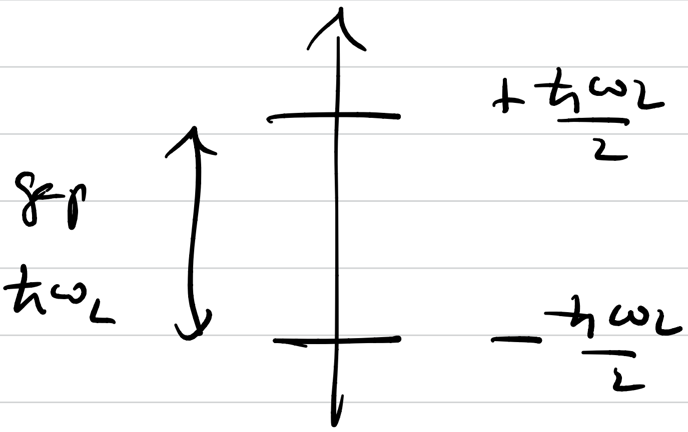
It is conventional to measure energies in terms of frequencies i.e we set $g B_z = \frac{\hbar \omega_L}{2}$ where ω_L is called the larmor frequency.

<u>Note</u> : $\omega_L \left[\frac{1}{s}\right]$ a [Hertz]

$$\hbar = [\text{Joules} \cdot \text{sec}] = \frac{\text{Planck const}}{2\pi} = \frac{h}{2\pi}$$

So
$$H = -\frac{\hbar \omega_L}{2} \sigma_z = \begin{pmatrix} -\frac{\hbar \omega_L}{2} & 0 \\ 0 & \frac{\hbar \omega_L}{2} \end{pmatrix}$$

This describes two energy levels (a two level syst)



$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ excited state or down spin

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ground state or up spin

Dynamics:

$$U_t = e^{-i\frac{tH}{\hbar}} = \begin{pmatrix} e^{\frac{it\omega_L}{2}} & 0 \\ 0 & e^{-i\frac{t\omega_L}{2}} \end{pmatrix}$$

initial state: $|\psi(0)\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle$

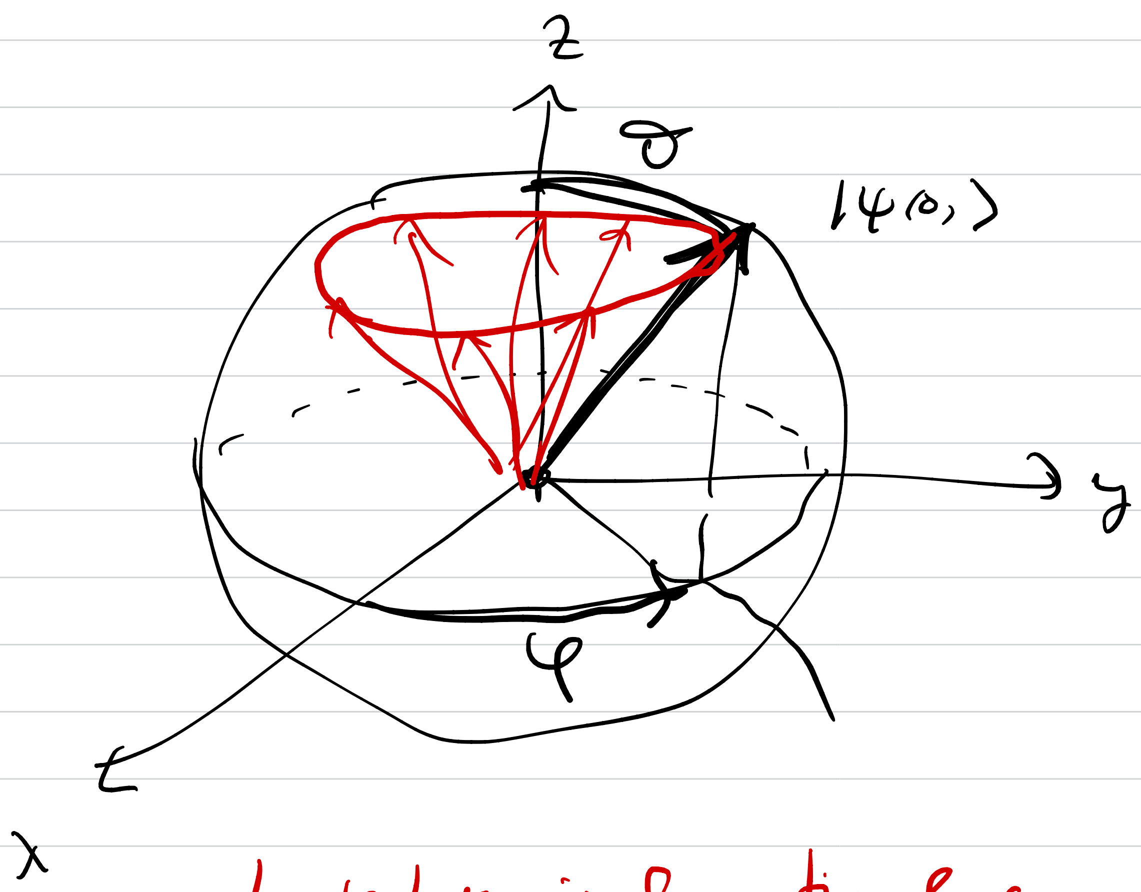
final state at time t:

$$|\psi(t)\rangle = U_t|\psi(0)\rangle$$

$$= \cos\frac{\theta}{2}\, e^{\frac{it\omega_L}{2}}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}\, e^{-\frac{it\omega_L}{2}}|\downarrow\rangle$$

$$|\psi(t)\rangle = \underbrace{e^{\frac{it\omega_L}{2}}}_{\substack{\text{global}\\\text{phase}\\\text{is unobservable}}} \underbrace{\left(\cos\frac{\vartheta}{2}|\uparrow\rangle + \left(\sin\frac{\vartheta}{2}\right)e^{i(\varphi - t\omega_L)}|\downarrow\rangle\right)}_{\text{time evolution.}}$$

<u>Bloch sphere representation:</u>



- trajectory is along tip of cone making an angle ϑ with z.
- <u>period</u> $\Rightarrow$ $T\omega_L = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega_L}}$
- This is called the <u>Larmor precession</u>

Dynamics in a time dependent magn field.

If $\vec{B}(t)$ depends on time and cannot anymore be taken along z it becomes difficult to compute exactly U_t in general. However for

$$\vec{B}(t) = \left((\cos \omega t) B_1, (\sin \omega t) B_1, B_z \right)$$

i.e rotating in xy plane we can compute exactly the dynamics.

$$H = -gB_1 (\cos \omega t \, \sigma_x + \sin \omega t \, \sigma_y) - g B_z \sigma_z.$$

set $gB_1 = \frac{\hbar \omega_1}{2}$

$gB_z = \frac{\hbar \omega_L}{2}.$

$$H = -\frac{\hbar\omega_1}{2}\left(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y\right) - \frac{\hbar\omega_L}{2}\sigma_z.$$

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = -i\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2}\right)$$

$$H = -\frac{\hbar\omega_1}{2}\left[\frac{(\sigma_x - i\sigma_y)}{2} e^{i\omega t} + \frac{(\sigma_x + i\sigma_y)}{2} e^{-i\omega t}\right] - \frac{\hbar\omega_L}{2}\sigma_z$$

$$\begin{cases} \frac{1}{2}(\sigma_x - i\sigma_y) \equiv \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \frac{1}{2}(\sigma_x + i\sigma_y) \equiv \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

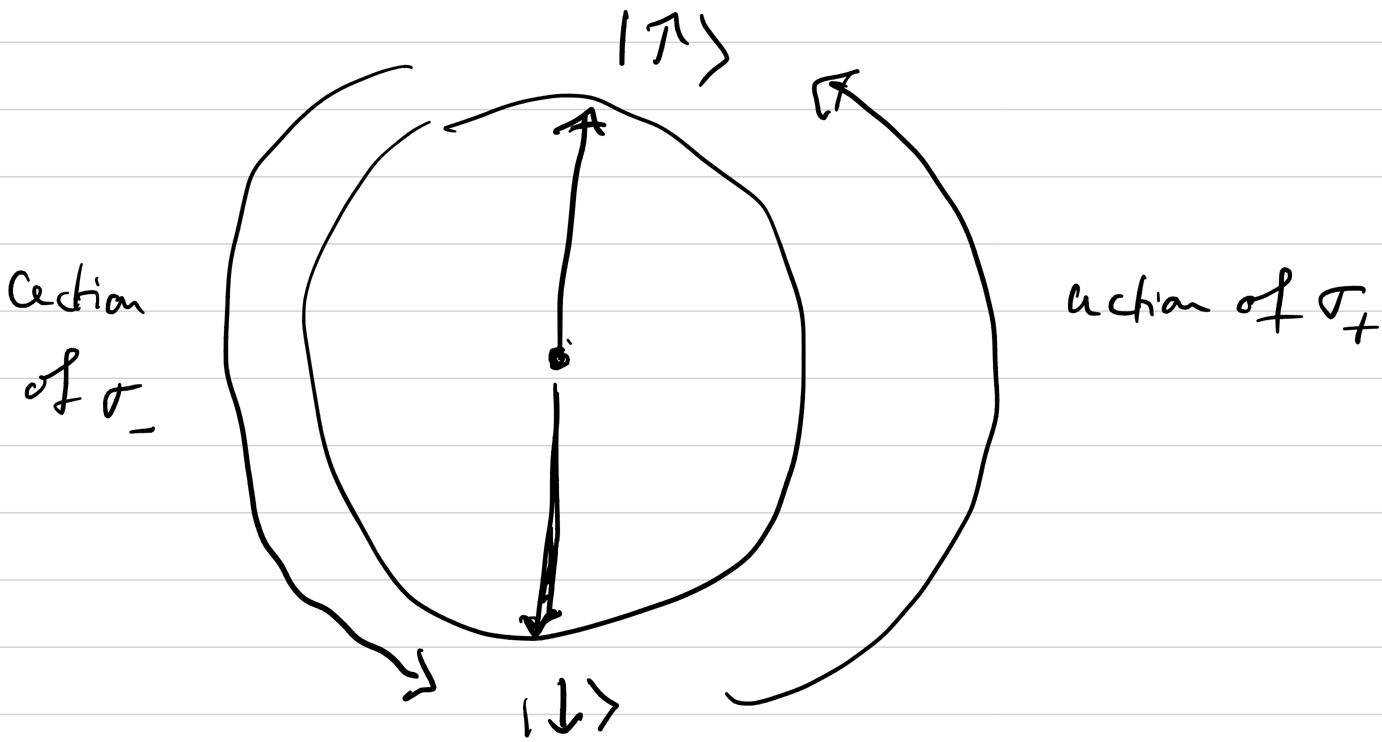
"ladder operators"

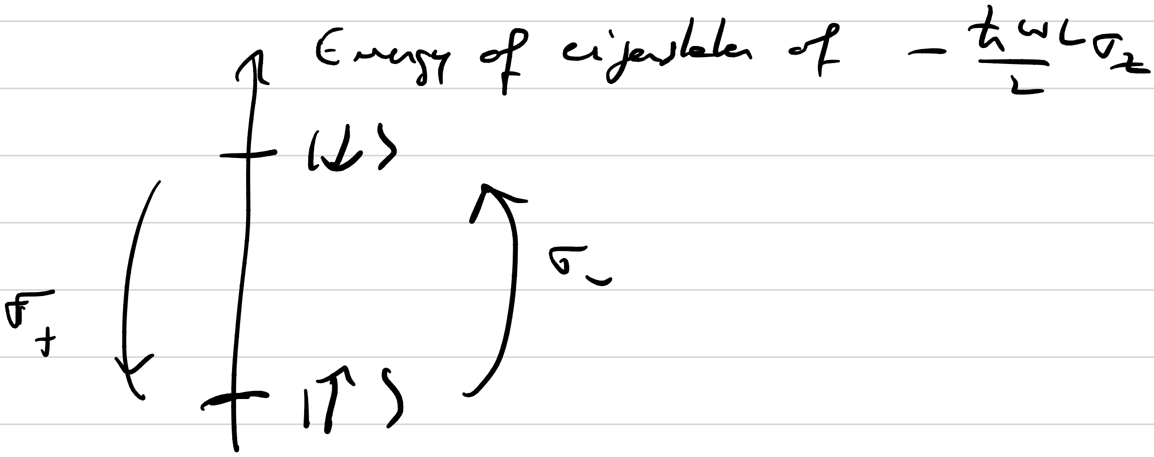
$$\boxed{H = -\frac{\hbar\omega_1}{2}\left[\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t}\right] - \frac{\hbar\omega_L}{2}\sigma_z.}$$

σ_+ and σ_- are called ladder operators. They

satisfy $\sigma_+|\uparrow\rangle = 0$, $\sigma_+|\downarrow\rangle = |\uparrow\rangle$

$\sigma_-|\uparrow\rangle = |\downarrow\rangle$, $\sigma_-|\downarrow\rangle = 0$ } exercise.





Unitary dynamics for :

$$H = -\frac{\hbar\omega_1}{2}\left[\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}\right] - \frac{\hbar\omega_2}{2}\sigma_z .$$

Note that here $H = H(t)$ depends on time.

⚠ it is not true that $\cancel{U_t = e^{\frac{it}{\hbar}H(t)}}$.

However U_t satisfies the Schroedinger equation

$$\boxed{i\hbar\frac{d}{dt}U_t = H(t)U_t \quad \text{or} \quad i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle}$$

It is not trivial how to solve it because $H(t)$ depends on time. However for above Hamiltonian there is an exact solution.

Here we give a heuristic derivation.

(Full derivation in exercises)

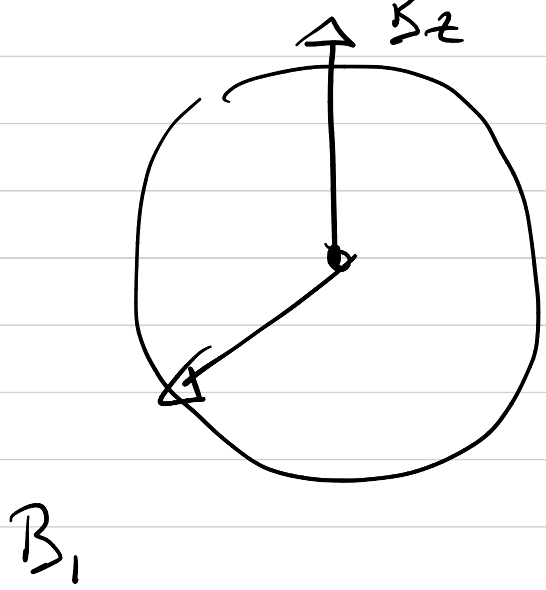
Since $\vec{B}(t) = (B_1 \cos\omega t, B_1 \sin\omega t, B_z)$

if we attach ourselves to a reference frame attached to rotating $\vec{B}(t)$ in xy plane we would see a different hamiltonian:

$$H_{\text{rotating frame}} = -\underbrace{\frac{\hbar\omega_1}{2}\sigma_x}_{} - \frac{\hbar}{2}\underbrace{(\omega_L - \omega)}_{}\sigma_z .$$

Magn field along x in rotating fram

$\delta =$ New Larmor precession frequency = "detuning parameter"



$$H_{\text{rot frame}} = \begin{pmatrix} -\frac{\hbar}{2}\delta & -\frac{\hbar\omega_1}{2} \\ -\frac{\hbar\omega_1}{2} & +\frac{\hbar\delta}{2} \end{pmatrix}.$$

One can show that $H(t)$ & $H_{\text{rot frame}}$ are related by a "similarity transf" (unitarily equiv).

$$U_t^{\text{rotating frame}} = e^{-\frac{it}{\hbar} H_{\text{rot frame}}}$$

time independent Hamiltonian

Now use

$$e^{i\alpha \hat{n}\cdot\vec{\sigma}} = (\cos\alpha)\mathbb{1} + i(\sin\alpha)\,\hat{n}\cdot\vec{\sigma}$$

for $\|\hat{n}\| = 1$ (exercise).

to compute the unitary evolution operator in the rotating frame exactly. (See exercises)

Here we discuss only special cases.

* Highly detuned situation :

$$\delta = \omega_L - \omega \gg \omega_1 .$$

$$\Rightarrow \quad H_{\text{rot fra}} \approx \begin{pmatrix} -\frac{\hbar\delta}{2} & 0 \\ 0 & \frac{\hbar\delta}{2} \end{pmatrix} .$$

$\Rightarrow$ { Dynamics given by Larmor precession δ
In lab frame this is ω_L .

Basically dynamics is roughly the one without B_1 .

* Tuned situation

$$\delta = \omega_L - \omega \ll \omega_1 \qquad (\text{or } \delta \simeq 0)$$

$$\Rightarrow H_{\text{rot frame}} \approx \begin{pmatrix} 0 & -\frac{\hbar\omega_1}{2} \\ -\frac{\hbar\omega_1}{2} & 0 \end{pmatrix} = -\frac{\hbar\omega_1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_{\text{rot frame}} \approx e^{i\frac{t\omega_1}{2}\sigma_x}$$

$$= \left(\cos\frac{t\omega_1}{2}\right)\mathbb{1} + i\left(\sin\frac{t\omega_1}{2}\right)\sigma_x$$

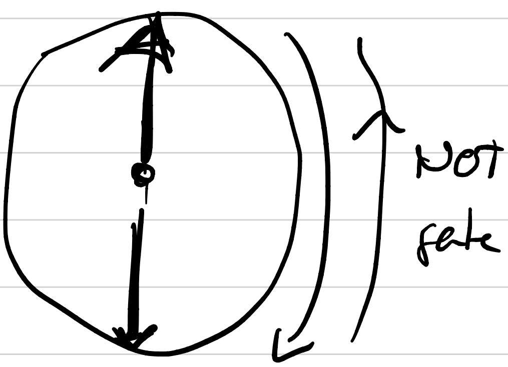
For $t = \frac{\pi}{\omega_1}$ $\quad U_{\text{rot frame}}\left(t = \frac{\pi}{\omega_1}\right) \approx i\sigma_x$

= "NOT gate"

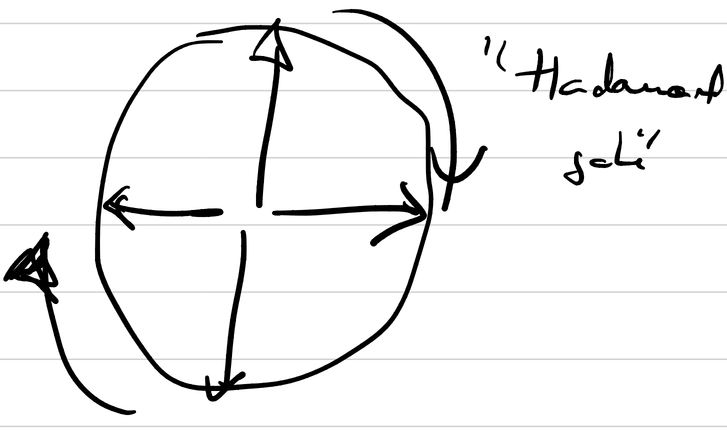
For $t = \frac{\pi}{2\omega_1}$ $\quad U_{\text{rot frame}}\left(t = \frac{\pi}{2\omega_1}\right) \approx \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\sigma_x\right)$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

= "Kind Hadamard gate"



time $\approx \frac{\pi}{\omega_1}$



time $\approx \frac{\pi}{2\omega_1}$.

8 Appendix: algebra of Pauli matrices.

It useful to use algebraic properties in calculations.

It is an exercise to check them.

1) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$

2) $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$ (x, y, z).

3) $[\sigma_x, \sigma_y] = 2i\sigma_z$ and cyclic permutations

$[\sigma_y, \sigma_z] = 2i\sigma_x$

$[\sigma_z, \sigma_x] = 2i\sigma_y$.

Note by def: $[A, B] = AB - BA$ the commutator.

4) $\exp(i\alpha\, \hat{n}\cdot\vec{\sigma}) = (\cos\alpha)\mathbb{1} + i(\sin\alpha)\, \hat{n}\cdot\vec{\sigma}$; $\|\hat{n}\| = 1$.